necessary that the group be transformable to an Abelian group. On the other hand, every group can be transformed by a change of variables to a group similar to it [2], and, by a suitable choice of base vectors, the similar groups will have like structure constants. According to the above-cited corollary there exists a change of coordinate systems transforming the first coordinate to an ignorable coordinate, but here the second coordinate becomes a latent ignorable coordinate.

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RESONANCE OSCILLATIONS OF A COMPOUND TORSION PENDULUM

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The oscillations of conservative systems with two degrees of freedom under internal resonance were examined in [1 - 6]. We investigate the resonance oscillations of one mechanical system and ascertain the features of its behavior.

1. Consider the system shown in Fig. 1. It consists of a disk attached to a thin elastic spindle having a coefficient of elasticity c. A compound pendulum rotates around an axis $O\eta$ belonging to the disk and perpendicular to the disk's axis of rotation (in the Figure this axis is perpendicular to the plane of the diagram). We take it that ξ, η, ζ are the principal inertial axes and that the compound pendulum has the principal moments of inertia I_ξ, I_η, I_ζ with respect to them. I is the disk's moment of inertia with respect to the axis of rotation. We denote the pendulum's center of gravity by C; the distance $OC = e$, φ_1 is the disk's angle of rotation from the equilibrium position, φ_2 is the pendulum's angle of deviation from the vertical, m is the mass of the pendulum. In this notation we have:

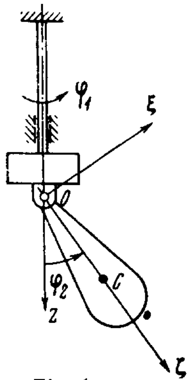

Fig. 1.

for the system's kinetic energy,

$$T = {}^1/_2 (I + I_\xi \sin^2 \varphi_2 + I_\varphi \cos^2 \varphi_2) \varphi_1^{\cdot 2} + {}^1/_2 I_\eta \varphi_2^{\cdot 2} \tag{1.1}$$

for the system's potential energy,

$$\Pi = {}^1/_2 c\varphi_1{}^2 - mge\sin\varphi_2 \tag{1.2}$$

Taking (1.1) and (1.2) into account, we find the equations of motion

$$\begin{gathered}I\varphi_1{}^{\cdot\cdot} + (I_\xi \sin^2\varphi_2 + I_\zeta \cos^2\varphi_2)\varphi_1{}^{\cdot\cdot} + (I_\xi - I_\zeta)\sin 2\varphi_2\varphi_1{}^{\cdot}\varphi_2{}^{\cdot} + c\varphi_1 = 0\\ I_\eta\varphi_2{}^{\cdot\cdot} - {}^1/_2(I_\xi - I_\zeta)\sin 2\varphi_2\varphi_1{}^{\cdot 2} + mge\sin\varphi_2 = 0\end{gathered} \tag{1.3}$$

We set

$$\varphi_1 = \varepsilon^{1/2} z_1, \qquad \varphi_2 = \varepsilon^{1/2} z_2 \qquad (\varepsilon \text{ is a small parameter}) \tag{1.4}$$

After some manipulations, from (1.3) we obtain

$$\begin{gathered}z_1'' + \beta^2 z_1 = -\varepsilon a(z_1'' z_2{}^2 + 2z_2 z_1' z_2') + \varepsilon^2 a({}^4/_3 z_2{}^3 z_1' z_2' + {}^1/_8 z_2{}^4 z_1'') + \ldots\\ z_2'' + z_2 = \varepsilon(b z_1'^2 z_2 + {}^1/_6 z_2{}^3) - \varepsilon^2({}^2/_3 b z_1'^2 z_2{}^3 + {}^1/_{120} z_2{}^5) + \ldots\end{gathered} \tag{1.5}$$

$$\tau = \left(\frac{mge}{I_\eta}\right)^{1/2} t, \qquad a = \frac{I_\xi - I_\zeta}{I + I_\zeta}, \qquad b = \frac{I_\xi - I_\zeta}{I_\eta}, \qquad \beta^2 = \frac{c}{I + I_\zeta}\frac{I_\eta}{mge} \tag{1.6}$$

The primes denote the derivatives with respect to a dimensionless time.

2. We seek the solution of system (1.5), using one supplement to the existing asymptotic methods [7]. Let us find the solution in the form

$$\begin{gathered}z_1 = A\cos(\beta\tau - \varphi) + \varepsilon z_{11} + \varepsilon^2 z_{12} + \cdots\\ z_2 = B\cos(\tau - \psi) + \varepsilon z_{21} + \varepsilon^2 z_{22} + \cdots\end{gathered} \tag{2.1}$$

where A, B, φ, ψ are slowly varying functions of τ, while $z_{11}, z_{21}, \ldots$ are additive corrections. Substituting (2.1) into (1.5) we find

$$\begin{gathered}(A'' + 2\beta A\varphi' - A\varphi'^2)\cos(\beta\tau - \varphi) + (A\varphi'' - 2\beta A' + 2A'\varphi')\sin(\beta\tau - \varphi) +\\ + \varepsilon(z_{11}'' + \beta^2 z_{11}) + \ldots = \varepsilon\{{}^1/_2 a\beta^2 AB^2\cos(\beta\tau - \varphi) +\\ + {}^1/_4 a\beta(\beta + 2)AB^2\cos[(\beta + 2)\tau - \varphi - 2\psi] +\\ + {}^1/_4 a\beta(\beta - 2)AB^2\cos\{(\beta - 2)\tau - \varphi + 2\psi]\}\end{gathered} \tag{2.2}$$

$$\begin{gathered}(B'' + 2B\psi' - B\psi'^2)\cos(\tau - \psi) + (B\psi'' - 2B' + 2B'\psi')\sin(\tau - \psi) + \varepsilon(z_{21}'' + z_{21}) + \ldots =\\ = \varepsilon\{({}^1/_2 b\beta^2 A^2 B + {}^1/_8 B^3)\cos(\tau - \psi) + {}^1/_{24} B^3\cos(3\tau - 3\psi) -\\ - {}^1/_4 b\beta^2 A^2 B\cos[(2\beta - 1)\tau - 2\varphi + \psi] -\\ - {}^1/_4 b\beta^2 A^2 B\cos[(2\beta + 1)\tau - 2\varphi - \psi]\}\end{gathered}$$

Here the terms containing ε to powers higher than the first have been discarded. If $\beta \neq 1$, the solution is nonresonant and is obtained without any special difficulties. For A and B we obtain constants defined by the initial conditions. Let us consider the resonant solution when $\beta \approx 1$ and $\beta = 1$.

We use the identities

$$\begin{gathered}\cos[(\beta - 2)\tau - \varphi + 2\psi] = \cos\lambda\cos(\beta\tau - \varphi) + \sin\lambda\sin(\beta\tau - \varphi)\\ \cos[(2\beta - 1)\tau - 2\varphi + \psi] = \cos\lambda\cos(\tau - \psi) - \sin\lambda\sin(\tau - \psi)\end{gathered} \tag{2.3}$$

$$\lambda = 2(\beta - 1)\tau - 2\varphi + 2\psi \tag{2.4}$$

From (2.2) we find the systems

$$\begin{gathered}A'' + 2\beta A\varphi' - A\varphi'^2 = \varepsilon{}^1/_2 a\beta^2 AB^2 - \varepsilon{}^1/_4 a\beta(2 - \beta)AB^2\cos\lambda\\ A\varphi'' - 2\beta A' + 2A'\varphi' = -\varepsilon{}^1/_4 a\beta(2 - \beta)AB^2\sin\lambda\\ B'' + 2B\psi' - 2\psi'^2 = \varepsilon({}^1/_2 b\beta^2 A^2 B + {}^1/_8 B^3) - \varepsilon{}^1/_4 b\beta^2 A^2 B\cos\lambda\\ B\psi'' - 2B' + 2B'\psi' = \varepsilon{}^1/_4 b\beta^2 A^2 B\sin\lambda\end{gathered} \tag{2.5}$$

$$\begin{gathered}z_{11}'' + \beta^2 z_{11} = {}^1/_4 a\beta(\beta + 2)AB^2\cos[(\beta + 2)\tau - \varphi - 2\psi]\\ z_{21}'' + z_{21} = -{}^1/_4 b\beta^2 A^2 B\cos[(2\beta + 1)\tau - 2\varphi - \psi] + {}^1/_{24} B^3\cos(3\tau - 3\psi)\end{gathered} \tag{2.6}$$

Equations (2.5) are called variational, while (2.6) are called the equations for the perturbations. Taking into account the slow variation of A, B, φ and ψ, from (2.6) we find

$$z_{11} = -{}^1/_{16}a\beta\frac{\beta+2}{\beta+1}AB^2\cos[(\beta+2)\tau-\varphi-2\psi] \tag{2.7}$$

$$z_{21} = {}^1/_{16}\frac{b\beta}{\beta+1}A^2B\cos[(2\beta+1)\tau-2\varphi-\psi] - {}^1/_{192}B^3\cos(3\tau-3\psi)$$

It is easily verified that any solution of the system

$$dA/\varepsilon d\tau = {}^1/_8 a\beta(2-\beta)AB^2\sin\lambda \qquad d\varphi/\varepsilon d\tau = {}^1/_4 a\beta B^2 - {}^1/_8 a\beta(2-\beta)B^2\cos\lambda$$
$$dB/\varepsilon d\tau = -{}^1/_8 b\beta A^2B\sin\lambda \qquad d\psi/\varepsilon d\tau = {}^1/_4 b\beta A^2 + {}^1/_{16}B^2 - {}^1/_8 b\beta A^2\cos\lambda \tag{2.8}$$

satisfies system (2.5) to within terms of second order in ε. From the first of the two equations in (2.8) we obtain, after an elimination of τ and an integration,

$$\sigma^2A^2 + B^2 = \sigma^2\varkappa^2 \qquad \left(\sigma^2 = \frac{b}{a(2-\beta)}\right) \tag{2.9}$$

Here $\varkappa^2$ is a constant of integration. From (2.8), (2.9) and (2.4) we obtain an autonomous system for the two variables A^* and λ

$$dA^*/du = A^*(1-A^{*2})\sin\lambda, \quad d\lambda/du = -2n + 4qA^{*2} + 2(1-2A^{*2})\cos\lambda \qquad \left(A^* = \frac{A}{\varkappa}\right)$$

$$u = {}^1/_8\beta b\varkappa^2\varepsilon\tau, \qquad 2n = -\frac{16}{b\beta\varkappa^2}\frac{\beta-1}{\varepsilon} + \frac{4a\beta-1}{b\beta}, \qquad 4q = \sigma^2\frac{4a\beta-1}{b\beta} - 4\beta \tag{2.10}$$

From (2.10) we find

$$[-2nA^* + 4qA^{*3} + 2A^*(1-2A^{*2})\cos\lambda]\,dA^* - A^{*2}(1-A^{*2})\sin\lambda d\lambda = 0 \tag{2.11}$$

This equation has the general integral

$$-nA^{*2} + qA^{*4} + A^{*2}(1-A^{*2})\cos\lambda = c_0 \tag{2.12}$$

Here c_0 is a constant of integration.

3. Let us investigate the phase trajectories for autonomous system (2.10) in the (x y)-plane for which $x = A^*\cos\lambda$, $y = A^*\sin\lambda$, i.e., A^* and λ are the natural polar coordinates. The phase trajectories are determined by expression (2.12) and all of them are symmetric relative to the axis Ox. In view of (2.9) all the real trajectories lie on the boundary or inside the circle $A^* = 1$.

In the first place we determine the singular points of system (2.10). From the conditions

$$dA^*/du = 0, \qquad d\lambda/du = 0 \tag{3.1}$$

we find the singular points

$$\begin{aligned} &\text{a)}\quad \lambda_1 = 0, && A_1^* = \left(\frac{1-n}{2(1-q)}\right)^{1/2} \\ &\text{b)}\quad \lambda_2 = \pi, && A_2^* = \left(\frac{1+n}{2(1+q)}\right)^{1/2} \\ &\text{c)}\quad \cos\lambda_3 = 2q-n, && A_3^* = 1 \end{aligned} \tag{3.2}$$

As a matter of fact the point (c) is really two singular points on the boundary of a circle located symmetrically relative to the x-axis. The origin also is a singular point since $dA^*/du = 0$ for $A^* = 0$. These three types of singular points depend upon the parameters q and n. They exist for certain values of parameters q and n, while for other values they do not. This connection becomes particularly clear for the points (a) and (b) if we write the second of Eqs. (3.1) in the form

$$n = f(x), \qquad f(x) = \begin{cases} 1 + 2(q-1)x^2 & (0 < x < 1) \\ -1 + 2(q+1)x^2 & (-1 < x < 0) \end{cases} \tag{3.3}$$

and we solve it graphically. In view of dependency (2.9) we consider only the interval $[-1, 1]$ for x. We see that the parameter q has one boundary value $q = 1$ which demarcates two types of parabola $z = f(x)$. Note, further, also the case $q = 0$ for which there is no term qA^{*4} in expression (2.12).

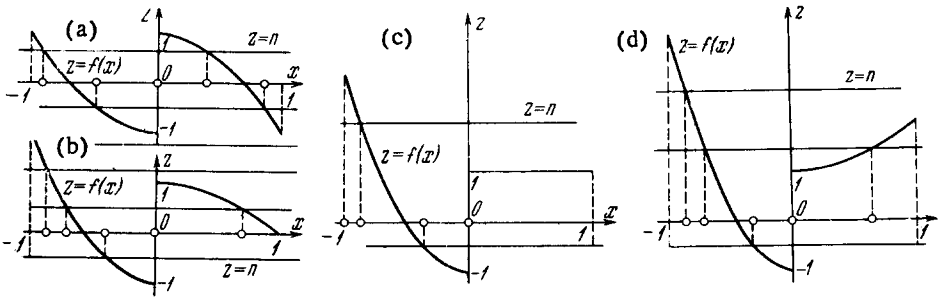

Fig. 2.

We consider the following cases: A. The value $q = 0$. B. The value of q is lower than the boundary value ($q = 0.5$). C. The boundary value $q = 1$. D. The value of q is higher than the boundary value ($q = 1.5$). The graphs of the function $z = f(x)$ for the four cases is shown in Fig. 2, a, b, c, d, respectively. We have drawn the straight lines $z = n$ parallel to the x-axis. The abscissas of the points of intersection of these straight lines with the curves $z = f(x)$ yield the desired singular points. As the straight line $z = n$ moves from $-\infty$ to $+\infty$ (i.e., for the interval $-\infty < n < +\infty$) it is very clear to see that when the singular points (a) and (b) exist and how they move along the x-axis.

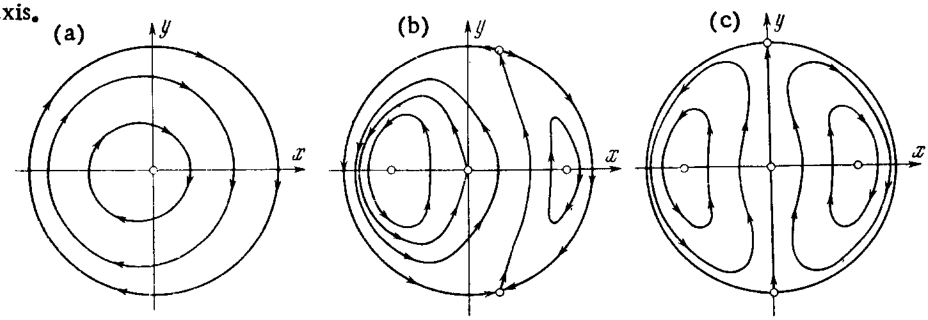

Fig. 3.

Let us consider the cases listed.

A. $q = 0$. Here the singular points (a), (b) and (c) exist in the interval $-1 \leqq n \leqq 1$ and they move to the left. The following subcases are distinguished as a function of n.

A.1. $n < -1$. Only one singular point exists, namely the reference origin, which is a center. The phase trajectories for such a case ($n = 1.5$) are shown in Fig. 3, a. We see that the amplitude A^* changes negligibly.

A.2. $-1 < n < 0$. Here exist all singular points (a), (b), which are centers, and points (c), which are saddles. The reference origin is a somewhat unusual singular point whose index equals zero. It can be looked upon as the result of a coalescence of a center and a saddle. The phase trajectories for such a case ($n = -0.25$) are shown in Fig. 3, b. We see that the amplitude A^* has a significant variation for certain phase trajectories.

A. 3. $n = 0$. This is a somewhat particular boundary case (Fig. 3, c). The phase trajectories are now further symmetric also relative to the y-axis. Conditions (3.1) are fulfilled for all points of the boundary circle, so that it can be treated as a singular line.

For a subsequent variation of n (for positive values) the phase trajectory patterns considered repeat, but in the reverse order with this difference that for some positive value of n the phase trajectories are the mirror images relative to the y-axis of the phase trajectories for that same negative value of n. For example, for $n = 0.25$ we have the phase trajectories of Fig. 3, b except that the x-axis must be turned to the left.

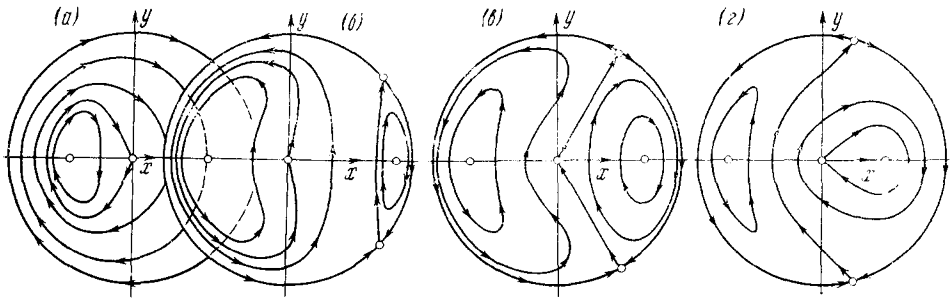

Fig. 4.

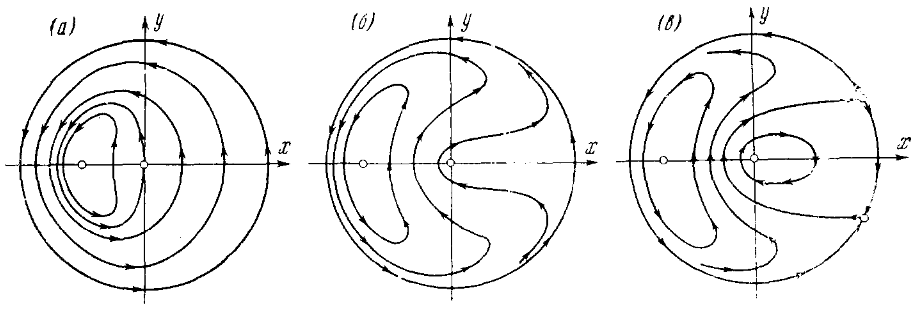

Fig. 5.

B. $q = 0.5$. Here the singular point (a) exists in the interval $-1 < n < 2$, the singular point (b), in the interval $0 < n < 1$, and singular points (c), in the interval $0 < n < 2$. Therefore, we distinguish the following subcases.

B. 1. $n < -1$. This subcase is analogous to A. 1. No singularities appear.

B. 2. $-1 < n < 0$. Here two singular points exist, (b), a center, and the origin whose index equals zero. The phase trajectories for such a case ($n = -0.25$) are shown in Fig. 4, a.

B. 3. $0 < n < 1$. All singular points exist here. In this subcase we can pick out three different behavior portraits of the phase trajectories.

B. 3. 1. $0 < n < 0.5$. The phase trajectory which passes through the reference origin is closed around the center (b). Such a case ($n = 0{,}25$) is shown in Fig. 4, b.

B.3.2. $n = 0.5$. This is a boundary case (Fig. 4, c). The phase trajectory passing through the origin is represented by two straight lines.

B.3.3. $0.5 < n < 1$. Here the phase trajectory which passes through the origin is closed around the center (a). Such a case ($n = 0.65$) is shown in Fig. 4, d.

B.4. $1 < n < 2$. Here, as also in B.2, there exist two singular points, (b) and the reference origin (0.0), only now the origin (0.0) is a center but point (b) is a singular point with an index equal to zero. We do not show the phase trajectories for this case, but they are analogous to C.4 (Fig. 5, c).

B.5. $2 < n$. Once again, as in B.1, we have only one singular point, namely the origin (0.0), which is a center.

C. $q = 1$ (boundary case). Here the singular point (a) does not exist. Point (b) exists in the interval $-1 < n < 3$ and the points (c), in the interval $1 < n < 3$. Therefore, we distinguish five subcases.

C.1. $n < -1$. This case is analogous to A.1 and B.1.

C.2. $-1 < n < 1$. Here two singular points (b) exist, one is a center, the other is the reference origin which is a singular point with an index equal to zero. Depending on the form of the trajectory passing through the origin, we can delineate three patterns.

C.2.1. $-1 < n < 0$. The phase trajectory passing through the origin and closing around center (b) has two tangents at the origin, which form an acute angle.

C.2.2. $n = 0$. The phase trajectory passing through the reference origin (Fig. 5, a) is tangent to the y-axis.

C.2.3. $0 < n < 1$. The phase trajectory passing through the origin is twice tangent to it, forming an obtuse angle. We do not show the phase trajectory patterns for cases C.2.1 and C.2.3, but they are analogous, respectively, to B.2 (Fig. 4, a) and D.2.3 (Fig. 6, a).

C.3. $n = 1$. This is a special boundary subcase. Here conditions (3.1) are fulfilled for all points of the x-axis in the interval $0 \leqq x \leqq 1$, so that this interval can be treated as a singular line. The phase trajectories for such a subcase are shown on Fig. 5, b. We see that for certain trajectories the amplitude has a significant variation.

C.4. $1 < n < 3$. Here exist the singular points (b) and the reference origin which are centers and the points (c) which are saddles. The phase trajectories for such a case ($n = 1.125$) are shown in Fig. 5, c.

C.5. $3 < n$. Here, as in B.5, we have only one singular point (0.0), a center.

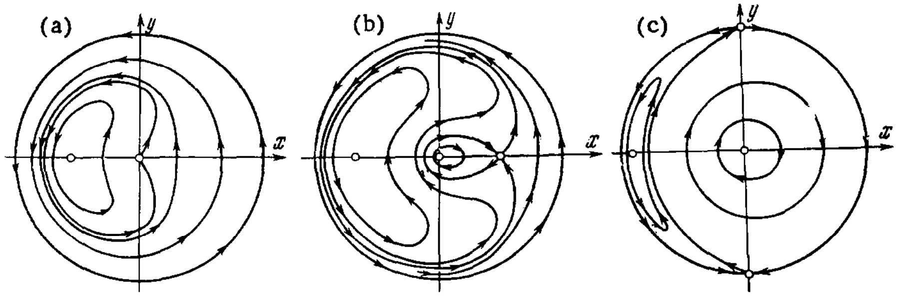

Fig. 6.

D. $q = 1.5$. Here (Fig. 2, d) the singular point (a) exists in the interval $1 < n < 2$, the point (b), in the interval $-1 < n < 4$, and the points (c), in the interval $2 < n < 4$.

Such a case was qualitatively investigated in [5]; it should be noted that the Cases A, B, C considered here were not observed there. Therefore, we shall not dwell on this case in detail. We have:

D.1. $n < -1$. This subcase is analogous to A.1, B.1, C.1.

D.2.1. $-1 < n < 0$. The pattern is analogous to C.2.1.

D.2.2. $n = 0$. The pattern is analogous to C.2.2.

D.2.3. $0 < n < 1$. The pattern is analogous (Fig.6, for $n < 0{,}5$)

D.3. $1 < n < 2$. This subcase differs essentially from the ones considered above. Here the point (a) is a saddle. Through it passes a separatrix a part of which envelops the center (b) while the other part envelops the other center, namely the reference origin (Fig. 6, b).

D.4. $2 < n < 4$. Here exist the singular points (b) and the reference origin which are centers and the points (c) which are saddles. This subcase is analogous to C.4. The phase trajectories for such a subcase are shown in Fig. 6, c.

D.5. $4 < n$. This subcase is analogous to B.5 and C.5.

If instead of $q = 1.5$ we take some other value of $q (q > 1)$ the qualitative results do not change. The phase trajectories have been considered for positive values of parameter q and for all values of parameter n. Negative values of parameter q were not considered since if in (2.12) we simultaneously reverse the signs of q, n, $\cos\lambda$, c_0 this equality is not altered. This shows that the investigations carried out for positive values of q carry over also for negative values of q if only we turn the x-axis to the left and replace n by $-n$. From (2.10) it follows further that we need also to turn around all the arrows on the phase trajectories. The phase trajectories allow us to analyze the system motion pattern. It is evident that motions with constant amplitude are possible; they correspond to points of the center type; and motions with periodic oscillations of amplitude are possible, which correspond to closed phase trajectories. Separatrices and singular saddle points correspond to periodic variations of the amplitude. In subcases A.2 - A.4, B.2 - B.4, C.2 - C.4, D.2 - D.4, for certain of the phase trajectories we can observe a transfer of energy from the rotating disk to the compound pendulum, where the amplitude of the torsion oscillations A^* decreases significantly while the amplitude of oscillations of the compound pendulum B increases significantly in view of dependency (2.9). Afterwards, A^* increases and B decreases. The energy is transferred from the compound pendulum to the rotating disk.

4. Let us further find the amplitude A^* as a function of the variable u. From (2.12) and (2.10) we obtain

$$\frac{d\theta}{\pm\sqrt{(1-\theta)^2\theta^2-(c_0+n\theta-q\theta^2)^2}} = 2du \qquad (\theta = A^{*2}) \tag{4.1}$$

Consider the polynomial

$$G(\theta) = (1-\theta)^2\theta^2 - (c_0 + n\theta - q\theta^2)^2 \tag{4.2}$$

The roots of polynomial (4.2) coincide with the roots (for A^{*2}) of Eq. (2.12) when $\cos\lambda = \pm 1$. For different values of n, q and c_0 polynomial (4.2) has four real roots or two complex and two real roots, i.e., it can be written in the form

$$G(\theta) = (1-q^2)(\theta-\theta_1)(\theta-\theta_2)(\theta-\theta_3)(\theta-\theta_4) \qquad (\theta_4 < \theta_3 < \theta_2 < \theta_1) \tag{4.3}$$

or in the form

$$G(\theta) = (1 - q^2)(\theta - \theta_1)(\theta - \theta_2)[(\theta - \nu)^2 + \omega^2] \qquad (0 < \theta_2 < \theta_1 < 1) \tag{4.4}$$

$$\nu = \frac{nq - 1}{q^2 - 1} - {}^1/_2 (\theta_1 + \theta_2) \qquad \omega^2 = \frac{c_0}{(q^2 - 1)\,\theta_1\theta_2} - \nu^2 \qquad (\omega > 0)$$

Every phase trajectory intersects the X-axis in two points, the squares of whose polar radii are the roots θ_i. The polynomial (4.2) is represented differently for the different cases (A,..., D).

In Cases A and B polynomial $G(\theta)$ has the form (4.3), but $\theta_4 < 0$,while $\theta_1 > 1$, i.e. real points in the (xy)-plane do hot correspond to them. Then, using [8], for θ in the interval $\theta_3 \leqq \theta \leqq \theta_2$ we obtain

$$\theta = \frac{\theta_3 (\theta_2 - \theta_4) - \theta_4 (\theta_2 - \theta_3) \operatorname{sn}^2 U}{\theta_2 - \theta_4 - (\theta_2 - \theta_3) \operatorname{sn}^2 U} \qquad \left(U = \frac{2\sqrt{1 - q^2}}{l} (u - u_0)\right) \tag{4.5}$$

Here the modulus k of the Jacobi elliptic function sn and the quantity l are determined by the expressions

$$k^2 = \frac{(\theta_1 - \theta_4)(\theta_2 - \theta_3)}{(\theta_1 - \theta_3)(\theta_2 - \theta_4)}, \qquad l^2 = \frac{4}{(\theta_1 - \theta_3)(\theta_2 - \theta_4)} \qquad (l > 0) \tag{4.6}$$

Here u_0 is the value of u when $\theta = \theta_3$.

The period of the long-period oscillations of the amplitude A^* with respect to time τ is determined by the formula

$$\varepsilon T = \frac{8l}{\beta b \varkappa^2 \sqrt{1 - q^2}} K(k) \tag{4.7}$$

Here $K(k)$ is the complete elliptic integral of the first kind in the Legendre form with modulus k.

In Case C (excepting the subcase C.3) the polynomial $G(\theta)$ has the form

$$G(\theta) = 2(n - 1)(\theta - \theta_1)(\theta - \theta_2)(\theta - \theta_3) \tag{4.8}$$

Here for $n > 1$ (subcases C.4, C.5) the root $\theta_1 > 1$, while for $n < 1$ (subcases C.1, C.2) the root $\theta_3 < 0$. Suppose $n > 1$, then using [8] we obtain for θ in the interval $\theta_3 \leqq \theta \leqq \theta_2$

$$\theta = \theta_3 + (\theta_2 - \theta_3) \operatorname{sn}^2 U_1 \qquad \left(U_1 = \frac{2\sqrt{2(n - 1)}}{l} (u - u_0)\right) \tag{4.9}$$

The modulus k of the Jacobi elliptic function sn and the quantity l are determined by the expressions

$$k^2 = \frac{\theta_2 - \theta_3}{\theta_1 - \theta_3}, \qquad l^2 = \frac{4}{\theta_1 - \theta_3} \qquad (l > 0) \tag{4.10}$$

Here u_0 is the value of u when $\theta = \theta_3$. The period of the long-period oscillations of the amplitude A^* with respect to time τ is determined by the formula

$$\varepsilon T = \frac{8l}{\beta b \varkappa^2 \sqrt{2(n - 1)}} K(k) \tag{4.11}$$

The modulus k and the quantity l are given by expressions (4.10).

Suppose $n < 1$. Using [8], for θ in the interval $\theta_2 \leqq \theta \leqq \theta_1$ we obtain

$$\theta = \frac{\theta_2 (\theta_1 - \theta_3) - \theta_3 (\theta_1 - \theta_2) \operatorname{sn}^2 U_2}{\theta_1 - \theta_3 - (\theta_1 - \theta_2) \operatorname{sn}^2 U_2} \qquad \left(U_2 = \frac{2\sqrt{2(1 - n)}}{l} (u - u_0)\right) \tag{4.12}$$

where the modulus k is determined by the expression

$$k^2 = (\theta_1 - \theta_2)/ (\theta_1 - \theta_3) \tag{4.13}$$

The quantity l is found from (4.10). The period of oscillations of the amplitude A^* is determined by the formula

$$\varepsilon T = \frac{8l}{\beta b\varkappa^2 \sqrt{2(1-n)}} K(k) \tag{4.14}$$

In the boundary case $n = 1$ (subcase C. 3) the polynomial $G(\theta)$ has the form

$$G(\theta) = 2c_0(\theta - \theta_1)(\theta - \theta_2)$$

and from (4.1) we easily obtain for θ in the interval $\theta_2 \leqq \theta \leqq \theta_1$

$$\theta = \theta_2 + (\theta_1 - \theta_2)\sin^2[\sqrt{-2c_0}(u - u_0)] \tag{4.15}$$

Here $c_0 < 0$ always and u_0 is the value of u for $\theta = \theta_2$.

In Case D we have $q > 1$, $1 - q^2 < 0$ and it is now possible to represent polynomial $G(\theta)$ in form (4.3) and in form (4.4). If $G(\theta)$ has form (4.3) the cases

(α) $\theta_4 < \theta_3 < 0 < \theta_2 < \theta_1 < 1$
(β) $0 < \theta_4 < \theta_3 < 1 < \theta_2 < \theta_1$
(γ) $0 < \theta_4 < \theta_3 < \theta_2 < \theta_1 < 1$

are possible. In case (γ) for an appropriate value of c_0 there exist two phase trajectories which intersect the x-axis at points with the polar radii $A_i^* = \sqrt{\theta_i}$, $i = 1,2,3,4$. In the first two cases, for an appropriate value of c_0 there exists only one phase trajectory which intersects the x-axis at points with polar radii $A_i^* = \sqrt{\theta_i}$ ($i = 1, 2$ for case (α), and $i = 3, 4$ for case (β)). For θ in the interval $\theta_2 \leqslant \theta \leqslant \theta_1$ we have

$$\theta = \frac{\theta_2(\theta_1 - \theta_3) - \theta_3(\theta_1 - \theta_2)\,\mathrm{sn}^2 U_3}{\theta_1 - \theta_3 - (\theta_1 - \theta_2)\,\mathrm{sn}^2 U_3} \qquad \left(U_3 = \frac{2\sqrt{q^2-1}}{l}(u - u_0)\right) \tag{4.16}$$

The modulus k of the elliptic function sn and the quantity l are determined by the expressions

$$k^2 = \frac{(\theta_3 - \theta_4)(\theta_2 - \theta_1)}{(\theta_3 - \theta_1)(\theta_2 - \theta_4)}, \qquad l^2 = \frac{4}{(\theta_1 - \theta_3)(\theta_2 - \theta_4)} \qquad (l > 0)) \tag{4.17}$$

Here u_0 is the value of u for $\theta = \theta_2$. For θ in the interval $\theta_4 \leqslant \theta \leqslant \theta_3$ we have

$$\theta = \frac{\theta_4(\theta_1 - \theta_3) + \theta_1(\theta_3 - \theta_4)\,\mathrm{sn}^2 U_3}{\theta_1 - \theta_3 + (\theta_3 - \theta_4)\,\mathrm{sn}^2 U_3} \tag{4.18}$$

Here k and l are the same as in (4.16), but u_0 is the value of u for $\theta = \theta_4$. For both intervals the oscillation period of the amplitude A^* is given by the expression

$$\varepsilon T = \frac{8e}{\beta b\varkappa^2 \sqrt{q^2 - 1}} K(k) \tag{4.19}$$

Suppose $G(\theta)$ has form (4.4). This can be realized only for phase trajectories enveloping the center (b) in subcase D. 3. Then we set [8]

$$\mathrm{tg}\, p_1 = \frac{\theta_1 - \nu}{\omega}, \quad \mathrm{tg}\, p_2 = \frac{\theta_2 - \nu}{\omega}, \quad \mu = \mathrm{tg}\,\frac{p_2 - p_1}{2}\,\mathrm{tg}\,\frac{p_2 + p_1}{2} \tag{4.20}$$

For θ in the interval $\theta_2 \leqslant \theta \leqslant \theta_1$ we have

$$\theta = \frac{\theta_1 + \theta_2}{2} - \frac{\theta_1 - \theta_2}{2}\,\frac{\mu - \mathrm{cn}\, U_3}{1 - \mu\,\mathrm{cn}\, U_3} \tag{4.21}$$

The modulus k of the Jacobi elliptic function cn and the quantity l are determined by the formulas

$$k^2 = \sin^2 \frac{p_1 - p_2}{2}, \qquad l = -\frac{(\cos p_1 \cos p_2)^{1/2}}{\omega} \tag{4.22}$$

Here u_0 is the value of u from $\theta = \theta_1$.The oscillation period of the amplitude has the form

$$\varepsilon T = \frac{16l}{\beta b \varkappa^2 \sqrt{q^2 - 1}} K(k) \tag{4.23}$$

Here k and l have the values (4.22).

After determining A^{*2} as a function of u we can also determine B^2 as a function of u by using (2.9). It should be noted that the resonance solution obtained for system (1.5) is valid also for the nonresonant case $\beta \neq 1$. From (2.10) we see that then $n \to \infty$ while from the cases considered it ensues that the phase trajectories are concentric circles, i.e., A and B are constants.

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